Locating and Total Dominating Sets of Direct Products of Complete Graphs

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Abstract

A set S of vertices in a graph $G = (V,E)$ is a metric-locating-total dominating set of G if every vertex of V is adjacent to a vertex in S and for every $u \neq v$ in V there is a vertex x in S such that $d(u,x) \neq d(v,x)$. The metric-location-total domination number $\gamma^M_t(G)$ of G is the minimum cardinality of a metric-locating-total dominating set in G. For graphs G and H, the direct product $G \times H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (x,y) and (v,w) are adjacent if and only if xv in $E(G)$ and yw in $E(H)$. In this paper, we determine the lower bound of the metric-location-total domination number of the direct products of complete graphs. We also determine some exact values for some direct products of two complete graphs.

Keywords: metric-locating-total dominating set, metric-location-total domination number, direct product

1 Introduction

Let $G = (V(G), E(G))$ be a graph with nonempty vertex set $V(G)$ and edge set $E(G)$. We consider only finite and simple graphs (without loops and multiple edges). We refer [2] for the general graph theory notations and terminologies are not described in this paper.

For an ordered set $W = \{w_1, w_2, ..., w_k\}$ of vertices and a vertex v in a connected graph G, the *representation of v with respect to W* is the ordered k-tuple $r(v|W) = (d(v,w_1), d(v,w_2), ..., d(v,w_k))$, where $d(x,y)$ represents the distance between the vertices x and y. The set W is called a *locating set* (LS) for G if every vertex of G has a distinct representation. A locating set containing a minimum number of vertices is called a basis for G. The metric dimension of G, denoted by $\dim(G)$, is the number of vertices in a basis of G.

It is easy to prove this following locating set property. We leave this lemma without proof.

Lemma 1 *Let $S \subset S' \subseteq V(G)$. If S_1 is a locating set, then S' is also locating.*

To determine whether W is a locating set for G, we only need to investigate the representations of the vertices in $V(G)\backslash W$, since the representation of each $w_i \in W$ has '0' in the ith-ordinate; and so it is always unique. If $d(u, x) \neq d(v, x)$, we shall say that vertex x *distinguishes* the vertices u and v and the vertices u and v *are distinguished* by x Likewise, if $r(u|S) \neq r(v|S)$, we shall say that the set S *distinguishes* vertices u and v.

Chartrand et.al [4] has characterized all graphs having metric dimensions 1, $n - 1$, and $n - 2$. They also determined the metric dimensions of some well known families of graphs such as paths, cycles, complete graphs, and trees. Caceres et.al in [1] stated the results of metric dimension of joint graphs. Caceres et.al in [3] investigated the characteristics of Cartesian product of graphs. Saputro et.al in [14] determined the metric dimension of Composition product of graphs.

A set $S \subseteq V(G)$ is a *dominating set* (DS) if each vertex in $V(G) – S$ is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. Similarly, $S \subseteq V(G)$ is a *total dominating set* (TDS) if each vertex in V is adjacent to at least one vertex of S. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set.

Let S be a TDS in a connected graph G. We call the set S a *metric-locating-total dominating set* (MLTDS) if S is also a locating set in G. We define the *metric-location-total domination number* $\gamma^M_t(G)$ of G to be the minimum cardinality of a MLTDS in G. A MLTDS in G of cardinality $\gamma^M_t(G)$ we call a $\gamma^M_t(G)$-set. The metric-location-total domination number is defined for every graph G with no isolated vertex, since V is such a set. Every MLTDS of a connected graph is a TDS and also a LS of the graph, so $\gamma_t(G) \leq \gamma^M_t(G)$ and $\dim(G) \leq \gamma^M_t(G)$ for every connected graph G. Therefore, it is easy to prove this following lemma.

Lemma 2 *For every connected graph,*

$$maks\{\gamma_t(G), \dim(G)\} \leq \gamma^M_t(G).$$

The *direct product* of G and H is the graph denoted by $G \times H$ (sometimes called *cross product*, *conjunction*, or *tensorial product*) with vertex set $V(G) \times V(H)$ where two vertices (x,y) and (v,w) are adjacent if and only if xv in $E(G)$ and yw in $E(H)$. The adjacency matrix of $G \times H$ is the tensor product of adjacency matrices of G and H.

Let $v \in H$. The subset $G_v = V(G) \times \{v\}$ is called the G-layer through v. The G-layers of direct product $G \times H$ are totally disconnected graphs on $|V(G)|$ vertices. Let $u \in G$. Similarly, we define H-layer through u is the subset $H_u = \{u\} \times V(H)$.

Imrich and Klavzar [9] stated that the direct product $G \times H$ is commutative and associative. Hence $G_1 \times G_2 \times \ldots \times G_k$ is well-defined. We denote the direct product of graphs $G_1 \times G_2 \times \ldots \times G_k$ as $\times^k_{i=1} G_i$.

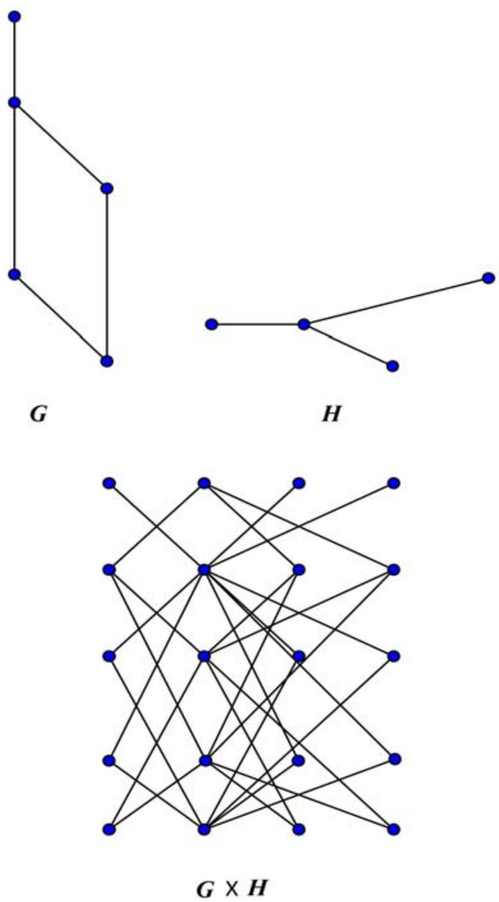

Figure 1. Graph *G*, graph *H* and their direct product graph.

Vizing [15] posed a well-known conjecture concerning the domination number of Cartesian product graphs

$$\gamma(G)\gamma(H) \leq \gamma(G \,\square\, \mathrm{H}).$$

30 years later, Gravier and Khelladi [7] posed an analogous conjecture for direct product graphs, namely

$$\gamma(G)\gamma(H) \leq \gamma(G \times H).$$

Nowakowski and Rall [12] and Klavzar and Zmamek [10] gave the counterexamples for the latter conjecture.

Rall [13], Zwierzchowski [16] and El-Zahar et al. [6] independently estimated the total domination of the direct product $G \times H$.

Theorem A [6, 13, 15] *For any G and H without isolated vertices holds*

$$\gamma_t(G \times H) \leq \gamma_t(G)\gamma_t(H).$$

Some exact values of the total domination number of direct products of certain graphs can be found in [5, 6, 16]. These results involve direct product of a cycle and a complete [5], paths and cycles [6], a path and a graph H without isolated vertices [16]. The survey of selected recent results on total domination in graphs can be found in Henning [8].

Imrich and Klavzar [9] assured the connectivity of direct product as stated in this following theorem.

Theorem B [9] *Let G and H be graphs with at least one edge. Then $G \times H$ is connected if and only if both G and H are connected and at least one of them is nonbipartite. Futhermore, if both G and H are connected and bipartite, then $G \times H$ has exactly two connected components.*

Since the complete graphs K_n, $n \geq 3$, is nonbipartite graph, then by using Theorem B, the $\times^t_{i=1} K_{n_i}$ is connected graph, for $n_i \geq 3$.

Mekis [11] stated a result on the domination and total domination of direct product of finitely many complete graphs. The result is stated in the following theorem.

Theorem C [11] *Let $G = \times^t_{i=1} K_{n_i}$, where $t \geq 3$ and $n_i \geq t + 1$ for all i. Then*

$$\gamma(G) = t + 1 = \gamma_t(G).$$

Mekis [11] also determined the exact values of domination number of some direct products of fewer than four complete graphs.

Theorem D [11] *For all $n_1, n_2, n_3 \in N$, $n_i \geq 2$,*

$$(i)\ \gamma\left(K_{n_1} \times K_{n_2}\right) = \begin{cases} 2, & n_i = 2 \text{ with } i \in \{1,2\} \\ 3, & \text{otherwise} \end{cases}$$

(ii) $\gamma\left(K_{n_1} \times K_{n_2} \times K_{n_3}\right) = 4$.

In the next section we will determine the metric-location-total-domination number of the direct products of complete graphs.

2 Metric-Location-Total Domination Number of Direct Products of Complete Graphs

For a complete graph K_n, $n \geq 1$, we assume that $V(K_n) = (1, 2, 3,\ldots, n)$. Let $\times_{i=1}^{t} K_{n_i}$ be the direct products of finitely many complete graphs. Since the degree of each vertex of K_{n_i} is $n_i - 1$ then by using the tensor product of adjacency matrices of K_{n_1}, K_{n_2}, …, K_{n_t} the degree of each vertex of $\times_{i=1}^{t} K_{n_i}$ is $(n_1 - 1)(n_2 - 1)\ldots(n_t - 1)$.

The vertices $u = (u_1, u_2, \ldots, u_t)$, $v = (v_1, v_2, \ldots, v_t) \in \times_{i=1}^{t} K_{n_i}$ are adjacent if and only if $u_i \neq v_i$ for i. Let $u \in K_{n_j}$. We define $\times_{\substack{i=1\\ i\neq j}}^{t} K_{n_i}$ -layer through u is the subset $V(K_{n_1}) \times \ldots \times V(K_{n_{j-1}}) \times \{u\} \times V(K_{n_{j+1}}) \times \ldots \times V(K_{n_t})$. For $u \in K_{n_1}$ or K_{n_t}, we denote $\times_{\substack{i=1\\ i\neq j}}^{t} K_{n_i}$ -layer through u as $\times_{i=2}^{t} K_{n_i}$ -layer through u or $\times_{i=1}^{t-1} K_{n_i}$ -layer through u respectively.

We start with the distance between two vertices in the direct products of finitely many complete graphs.

Lemma 2 *Let* $u, v \in \times_{i=1}^{t} K_{n_i}$ *with* $n_i \geq 3$ *for all* $i \in \{1, 2, \ldots, t\}$*, Then*

$$d(u,v) = \begin{cases} 1, & uv \in E\left(\times_{i=1}^{t} K_{n_i}\right) \\ 2, & uv \notin E\left(\times_{i=1}^{t} K_{n_i}\right) \end{cases}$$

Proof. Case 1 $uv \in E\left(\times_{i=1}^{t} K_{n_i}\right)$. It is obvious $d(u,v) = 1$.

Case 2. $uv \notin E\left(\times_{i=1}^{t} K_{n_i}\right)$. Let $u = (u_1, u_2, \ldots, u_t)$, $v = (v_1, v_2, \ldots, v_t) \in \times_{i=1}^{t} K_{n_i}$ where $u_k = v_k$ for some $k \in \{1, 2, \ldots, t\}$ and $u_l \neq v_l$ for some $l \in \{1, 2, \ldots, t\}$. Since $n_i \geq 3$ for all $i \in \{1, 2, \ldots, t\}$ then for each $l \in \{1, 2, \ldots, t\}$ there exist $w_l \neq u_l = v_l$ and for each $k \in \{1, 2, \ldots, t\}$ there exist $w_k \neq u_k \neq v_k$ such that $d(u, v) = d((u_1, u_2, \ldots, u_t), (v_1, v_2, \ldots, v_t)) = d((u_1, u_2, \ldots, u_t), (w_1, w_2, \ldots, w_t)) + d((w_1, w_2, \ldots, w_t), (v_1, v_2, \ldots, v_t)) = 1 + 1 = 2$. □

Now, we will determine the lower bound of the metric-location-total domination number of direct products of finitely many complete graphs.

Theorem 1 *Let* $G = \times_{i=1}^{t} K_{n_i}$ *be the direct products of complete graphs, where* $t \geq 3$ *and* $n_i \geq t + 3$ *for all i. Then*

$$dim(G) \geq t + 2.$$

Proof. Suppose that $W = \{w_1, w_2, \ldots, w_{t+1}\}$ is a basis of G (of size $t + 1$). Since $t \geq 3$ and $n_i \geq t + 3$ for all i, then there are two layers through u and v, where u and $v \in K_{n_i}$ for all i such that the intersection of vertices of each those layers and W is empty. Without loss of generality, the two layers are $\times_{i=2}^{t} K_{n_i}$ -layer through 1 and $\times_{i=2}^{t} K_{n_i}$ -layer through 2. Let $x = (1, x_2^{(1)}, \ldots, x_t^{(1)}) \in \times_{i=2}^{t} K_{n_i}$ -layer through 1 and $y = (2, y_2^{(2)}, \ldots, y_t^{(2)}) \in \times_{i=2}^{t} K_{n_i}$ -layer through 2 where $x_2^{(1)} = y_2^{(2)}$, …, $x_t^{(1)} = y_t^{(2)}$. Since every vertex $z \notin \times_{i=2}^{t} K_{n_i}$ -layer through 1 and $\times_{i=2}^{t} K_{n_i}$ -layer through 2, $d(x, z) = (1, \ldots) = d(y, z)$, then , $d(x, w) = d(y, w)$ for every $w \in W$. Therefore, $r(x|W) = r(y|W)$, a contradiction. □

Corollary 1 *Let* $G = \times_{i=1}^{t} K_{n_i}$ *be the direct products of complete graphs, where* $t \geq 3$ *and* $n_i \geq t + 3$ *for all i. Then*

$$\gamma^{M}_{t}(G) \geq t + 2.$$

Proof. It is easy to prove this corollary by using Lemma 2, Theorem C, and Theorem 1. □

We will determine the exact values of metric-location-total domination number of some direct products of two complete graphs. For $K_{n_i} \times K_{n_i}$ and $i \in \{1, 2, \ldots, n_i\}$, we label the vertices in K_{n_i} -layer through i as $\{(i, 1), (i, 2), \ldots, (i, n_i)\}$. We called the vertex (i, j) in K_{n_i} -layer through i as the j-th vertex in K_{n_i} -layer through i.

Proposition 1

(i) $\gamma^{M}_{t}(K_3 \times K_2) = 4$.

(ii) $\gamma^{M}_{t}\left(K_{n_i} \times K_{n_i}\right) = n_i$*, where* $n_i \geq 3$ *for all i.*

Proof. (i) The direct product $K_3 \times K_2$ is isomorphic to cycle C_6. From Chartrand [4], we know $\dim(C_6) = 2$ and $\gamma_t(C_6) = 4$ (by checking). By using Lemma 1, we can construct a total dominating set D from a resolving set S of C_6. Therefore, $\gamma^{M}_{t}(K_3 \times K_2) = 4$.
(ii) We will show that $\gamma^{M}_{t}\left(K_{n_i} \times K_{n_i}\right) \geq n_i$. Assume

on the contrary that $S = \{s_1, s_2, \ldots, s_{n_i-1}\}$ is a resolving set of $K_{n_i} \times K_{n_i}$ (of size n_i - 1). By using the similar reason shown in the prove of Theorem 1, we conclude that at most one of K_{n_i} -layer through i is the subset of $K_{n_i} \times K_{n_i}$ such that the intersection of this subset and any resolving set is empty. Let $V(K_{n_i}$ -layer through $n_i) \cap S = \emptyset$. It means the intersection of K_{n_i} -layer through i and S is exactly one vertex for every $i \in \{1, 2, \ldots, n_i - 1\}$. Let $V(K_{n_i}$ -layer through $i) \cap S = s_i$, for every $i \in \{1, 2, \ldots, n_i - 1\}$. Then, we label $s_i = (i, x_i)$ where $x_i \in \{1, 2, \ldots, n_i\}$.

Case 1. All of x_i are different. Without loss of generality, we arrange the set S such that $S = \{(1, 1), (2, 2), \ldots, (n_i - 1, n_i - 1)\}$. Consider vertex $u = (1, 2) \in K_{n_i}$ -layer through 1 and $v = (2, 1) \in K_{n_i}$ -layer through 2. Then for every $x \in \{u, v\}$, $d(x, (i, i)) = 2$ if $i = 1$ and 2 and $d(x, (i, i)) = 1$ if $i \in \{3, 4, \ldots, n_i - 1\}$. Therefore, $r(u|S) = r(v|S)$, a contradiction.

Case 2. There are $x_i = x_j$ for $i \neq j$. Without loss of generality, suppose that (1, 1), (2, 1) ∈ S. Consider vertex $u = (1, 2)$ and $v = (1, n_i)$ in K_{n_i} -layer through 1. Then, we also have $r(u|S) = r(v|S)$, a contradiction. Therefore, $\dim\left(K_{n_i} \times K_{n_i}\right) \geq n_i$. Then, by using Lemma 2, we can conclude that $\gamma^M{}_t\left(K_{n_i} \times K_{n_i}\right) \geq n_i$.

Next, we will prove $\gamma^M{}_t\left(K_{n_i} \times K_{n_i}\right) \leq n_i$. Set $S = \{(1, 1), (2, 2), \ldots, (n_i, n_i)\}$. We can easily prove the set S is a dominating set and a resolving set of $K_{n_i} \times K_{n_i}$. Additionally, S induces a complete graph on n_i vertices. Then, $\gamma^M{}_t\left(K_{n_i} \times K_{n_i}\right) \leq n_i$. □

3 Conclusions

This paper present that the lower bound of the metric-location-total domination number of the direct products of complete graphs does not depend on the number of vertices. But, some exact values for some direct products of two complete graphs depend on the number of vertices.

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