The Metric Dimension of Graph with Pendant Edges

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Abstract

For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph G, the representation of v with respect to W is the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ where $d(x, y)$ represents the distance between the vertices x and y. The set W is called a resolving set for G if every two vertices of G have distinct representations. A resolving set containing a minimum number of vertices is called a basis for G. The dimension of G, denoted by $\dim(G)$, is the number of vertices in a basis of G. In this paper, we determine the dimensions of some corona graphs $G \odot K_1$, $G \odot \overline{K_m}$, for any graph G and $m \geq 2$, and a graph with pendant edges more general than corona graphs $G \odot \overline{K_m}$.

1 Introduction

In this paper we consider finite, simple, and connected graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We refer the general graph theory notations and terminologies are not described in this paper to the book *Graphs and Digraphs* [6].

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$ of vertices, we refer to the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ as the (*metric*) *representation of v with respect to* W. The set W is called a *resolving set* for G if $r(u|W) = r(v|W)$ implies that $u = v$, for all $u, v \in G$. A resolving set of minimum

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cardinality for a graph G is called a *minimum resolving set* or a *basis* for G. The *metric dimension* $\dim(G)$ is the number of vertices in a basis for G.

The beginning papers for the idea of a resolving set (and of a minimum resolving set) were written by Slater in [10] and [11]. Slater introduced the concept of a resolving set for a connected graph G under the term *location set*. He called the cardinality of a minimum resolving set the *location number* of G. Independently, Harary and Melter [8] introduced the same concept but used the term *metric dimension*, rather than location number.

Chartrand et. al. [5] determined the bounds of the metric dimensions for any connected graphs and determined the metric dimensions of some well known families of graphs such as trees, paths, and complete graphs. Buczkowski et. al. [1] stated the existence of a graph G with $\dim(G) = k$ or a k-dimensional graph, for every integer $k \geq 2$. They also determined dimensions of wheels. Chappell et. al. [4] considered relationships between metric dimension with other parameters in a graph. Another researchers in [2, 7] determined the metric dimension of cartesian products of graphs and Cayley digraphs. In the following, we present some known results.

Theorem A ([2, 7])**.** *Let G be a connected graph of order $n \geq 2$.*

(*i.*) *$dim(G) = 1$ if and only if $G = P_n$*

(*ii.*) *$dim(G) = n - 1$ if and only if $G = K_n$*

(*iii.*) *For $n \geq 3$, $dim(C_n) = 2$*

(*iv.*) *For $n \geq 4$, $dim(G) = n - 2$ if and only if $G = K_{r,s}, (r, s \geq 1)$, $G = K_r + \overline{K_s}$, $(r \geq 1, s \geq 2)$, or $G = K_r + (K_1 \cup K_s)$, $(r, s \geq 1)$*

(*v.*) *If T is a tree that is not a path, then $dim(T) = \sigma(T) - ex(T)$, where $\sigma(T)$ denotes the sum of the terminal degrees of the major vertices of T, and $ex(T)$ denotes the number of the exterior major vertices of T.*

Let G and H be two given graphs with G having n vertices, the *corona product* $G \odot H$ is defined as a graph with

$$V(G \odot H) = V(G) \cup \bigcup_{i \in V(G)} V(H_i),$$
$$E(G \odot H) = E(G) \cup \bigcup_{i \in V(G)} (E(H_i) \cup \{iu_i | u_i \in V(H_i)\}),$$

where $H_i \cong H$, for all $i \in V(G)$. If $H \cong \overline{K_m}$, $G \odot H$ is equal to the graph produced by adding n pendant edges to every vertex of G. Especially, if $H \cong K_1$, $G \odot H$ is equal to the graph produced by adding one pendant edge to every vertex of G. Buczkowski et. al. in [1] proved that if G' is a graph obtained by adding a pendant edge to a nontrivial connected graph G, then

$$\dim(G) \leq \dim(G') \leq \dim(G) + 1.$$

Therefore, for $G \odot K_1$ we have:

$$\dim(G) \leq \dim(G \odot K_1).$$

If $G \cong K_1$ and $H \cong C_n$, $G \odot H$ is equal to wheel $W_n = K_1 + C_n$. If $G \cong K_1$ and $H \cong P_n$, $G \odot H$ is equal to fan $F_n = K_1 + P_n$. Buczkowski et. al. and Caceres et. al. in [1, 3], determined the dimensions of wheels and fans, namely: $\dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$, for $n \notin \{3, 6\}$, and $\dim(F_n) = \lfloor \frac{2n+2}{5} \rfloor$, for $n \notin \{1, 2, 3, 6\}$.

Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two graphs. The *Cartesian products* $G_1 \times G_2$ is the new graph whose vertex set is $V_1 \times V_2$ and two vertices (x_1, x_2) and (y_1, y_2) being adjacent in $G_1 \times G_2$ if and only if either $x_1 = y_1$ and $x_2y_2 \in E_2$ or $x_2 = y_2$ and $x_1y_1 \in E_1$. K_1 or P_1 is a *unit* with respect to the Cartesian product. In other words, $H \times G = G$ and $G \times H = G$ for any graph G, with $H = K_1$ or P_1. Caceres et. al. [3] determined the metric dimension of some cartesian product graphs, namely: $\dim(P_m \times P_n) = 2$, $\dim(P_m \times K_n) = n - 1$, for $n \geq 3$, and

$$\dim(P_m \times C_n) = \begin{cases} 2, & \text{if } n \text{ odd;} \\ 3, & \text{if } n \text{ even } (m \neq 1). \end{cases}$$

In this paper, we determine the dimensions of some corona graphs in $G \odot K_1$ and $G \odot \overline{K_m}$, for any graph G and $m \geq 2$. We also consider the dimension of a graph with pendant edges more general than corona graphs $G \odot \overline{K_m}$ obtained from graph G by adding a (not necessarily the same) number of pendant edges to every vertex of G.

2 Results

In Theorem 1, we will determine the metric dimension of $C_n \odot K_1$. This class of graph is known as the *sun* graphs Sun(n). Let $B = \{w_1, w_2, \cdots, w_k\}$ is a basis of Sun(n), v is a vertex in G and u is a pendant vertex of v in Sun(n). If the representation of vertex $v \in$ Sun(n) by B is $r(v|B) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$, then $r(u|B) = (d(v, w_1) + 1, d(v, w_2) + 1, \cdots, d(v, w_k) + 1)$, for $u \notin B$. It is easy to show that dim(Sun(3)) = dim(Sun(4)) = dim(Sun(5)) = 2, and dim(Sun(6)) = 3. For $n \geq 7$, the dimension of Sun(n) is 2 for odd n and 3 for even n.

Theorem 1. *For $n \geq 7$,*

$$dim(Sun(n)) = \begin{cases} 2, & n \ is \ odd, \\ 3, & n \ is \ even. \end{cases}$$

Proof Let $\mathrm{Sun}(n) = C_n \odot K_1$, where $C_n : v_1, v_2, \cdots, v_n$, and let u_i is a pendant vertex of v_i, for $n \geq 7$.

Case 1 $n = 2l + 1$ for some integer $l \geq 3$. First, we show $\dim(\mathrm{Sun}(n)) \leq 2$ by constructing a resolving set in $\mathrm{Sun}(n)$ with 2 vertices. Choose a resolving set $B = \{u_1, u_l\}$. The representation of vertices v's by B are

$r(v_k|B) = (k, l - k + 1)$, for $1 \leq k \leq l$,
$r(v_{l+1}|B) = (l + 1, 2)$,
$r(v_k|B) = (n - k + 2, k - l + 1)$, for $l + 2 \leq k \leq n - 1$,
$r(v_n|B) = (2, l + 1)$.

By inspection directly, for every pair u and $v \in V(\mathrm{Sun}(n)) - B$, and $u \neq v$, $r(u|B) \neq r(v|B)$. So, B is a resolving set. Then, by using Theorem A (i), $\dim(\mathrm{Sun}(n)) = 2$.

Case 2 $n = 2l$ for some integer $l \geq 4$. We will show that $\dim(\mathrm{Sun}(n)) \geq 3$. By Theorem A (i), we only need show that $\dim(\mathrm{Sun}(n)) \neq 2$. Suppose that $\dim(\mathrm{Sun}(n)) = 2$. Let $B = \{x, y\}$ is a resolving set of $\mathrm{Sun}(n)$.

Subcase 2.1 $x, y \in \{v_1, v_2, \cdots, v_n\}$. By symmetry, we can assume that ($x = v_1$ and $y = v_{l+1}$) or ($x = v_1$ and $y = v_k$, with $2 \leq k \leq l$). If $x = v_1$ and $y = v_{l+1}$ then $r(v_2|B) = (1, l-1) = r(v_n|B)$, a contradiction. If $x = v_1$ and $y = v_k$, $2 \leq k \leq l$, then $r(u_k|B) = (k, 1) = r(v_{k+1}|B)$, a contradiction.

Subcase 2.2 $x, y \in \{u_1, u_2, \cdots, u_n\}$. Again by symmetry, if $x = u_1$ and $y = u_k$, with $2 \leq k \leq l - 1$, then $r(u_{k+1}|B) = (k + 2, 3) = r(v_{k+2}|B)$, a contradiction. If $x = u_1$ and $y = u_l$ then $r(u_{l-1}|B) = (l, 3) = r(v_{l+2}|B)$, a contradiction. If $x = u_1$ and $y = u_{l+1}$ then $r(v_2|B) = (2, l) = r(v_n|B)$, a contradiction.

Subcase 2.3 $x \in \{u_1, u_2, \cdots, u_n\}$ and $y \in \{v_1, v_2, \cdots, v_n\}$ or reverse. Let be the previous one. If $x = u_1$ and $y = v_1$ then $r(v_2|B) = (2, 1) = r(v_n|B)$, a contradiction. If $x = u_1$ and $y = v_k$, with $2 \leq k \leq l$, then $r(u_k|B) = (k + 1, 1) = r(v_{k+1}|B)$, a contradiction. If $x = u_1$ and $y = v_{l+1}$, then $r(v_2|B) = (2, l - 1) = r(v_n|B)$, a contradiction.

Therefore, $\dim(\mathrm{Sun}(n)) \geq 3$. Next, we will show that $\dim(\mathrm{Sun}(n)) \leq 3$. Choose a resolving set $B = \{u_1, u_2, u_l\}$, then the representation of vertices $v \in C_n$ by B are

$r(v_1|B) = (1, 2, l)$,
$r(v_k|B) = (k, k - 1, l - k + 1)$, for $2 \leq k \leq l$,
$r(v_{l+1}|B) = (l + 1, l, 2)$,

$r(v_k|B) = (n-k+2, n-k+3, k-l+1)$, for $l+2 \leq k \leq n$.
By inspection directly, for every pair u and $v \in V(\text{Sun}(n)) - B$ and $u \neq v$, $r(u|B) \neq r(v|B)$. Therefore, B is a resolving set, and so $\dim(\text{Sun}(n)) = 3$. ■

In the next theorem, the dimension of $(P_n \times P_m) \odot K_1$ will be discussed. For small numbering n and m, we have $\dim((P_1 \times P_1) \odot K_1) = \dim(P_2) = 1$, $\dim((P_2 \times P_1) \odot K_1) = \dim(P_2) = 1$, and $\dim((P_2 \times P_2) \odot K_1) = \dim(\text{Sun}(n)) = 2$.

Theorem 2. *For $n \geq 3$ and $1 \leq m \leq 2$, $dim((P_n \times P_m) \odot K_1) = 2$.*

Proof Let $v_{ij} = (v_i, v_j)$ be the vertices of $P_n \times P_m \subseteq (P_n \times P_m) \odot K_1$, where $v_i \in P_n$, $v_j \in P_m$, $1 \leq i \leq n$, and $1 \leq j \leq m$. Let u_{ij} be the pendant vertex of v_{ij}.

Case 1. $m = 1$. By using Theorem A (i), we only need to show that $\dim((P_n \times P_1) \odot K_1) \leq 2$. Choose a resolving set $B = \{v_{11}, v_{n1}\}$ in $(P_n \times P_1) \odot K_1$. The representation of vertices $v \in (P_n \times P_1) \odot K_1$ by B are

$r(v_{i1}|B) = (i-1, n-i)$, for $2 \leq i \leq n-1$,

$r(u_{i1}|B) = (d(v_{11}, v_{i1}) + 1, d(v_{n1}, v_{i1}) + 1)$, for $1 \leq i \leq n$.

All of those representation are distinct. Therefore, $\dim((P_n \times P_1) \odot K_1) = 2$.

Case 2. $m = 2$. Again, by Theorem A (i), we only need to show that $\dim((P_n \times P_2) \odot K_1) \leq 2$. Choose a resolving set $B = \{u_{11}, u_{12}\}$ in $(P_n \times P_2) \odot K_1$. The representation of vertices $v \in (P_n \times P_2) \odot K_1$ by B are

$r(v_{i1}|B) = (i, i+1)$ and $r(v_{i2}|B) = (i+1, i)$, for $1 \leq i \leq n$,

$r(u_{i1}|B) = (d(v_{i1}, u_{11}) + 1, d(v_{i1}, u_{12}) + 1)$

and $r(u_{i2}|B) = (d(v_{i2}, u_{11}) + 1, d(v_{i2}, u_{12}) + 1)$, for $2 \leq i \leq n$.

All of those representation are distinct. Therefore, $\dim((P_n \times P_2) \odot K_1) = 2$. ■

Open problem 1. *Find the dimension of $(P_n \times P_m) \odot K_1$, for $n \geq 3$ and $m \geq 3$.*

Theorem 3. *For $n \geq 3$ and $1 \leq m \leq 2$*

$$dim((K_n \times P_2) \odot K_1) = \begin{cases} n-1, & m = 1, \\ n, & m = 2. \end{cases}$$

Proof Let $v_{ij} = (v_i, v_j)$ is a vertex in $K_n \times P_2$, where $v_i \in K_n$, $v_j \in P_2$, $1 \leq i \leq n$, and $1 \leq j \leq 2$. Let u_{ij} is a pendant vertex of v_{ij}.

Case 1. $m = 1$. By a contradiction, we show $\dim((K_n \times P_1) \odot K_1) \geq n-1$. Suppose that B is a basis of $(K_n \times P_1) \odot K_1$ with $|B| < n-1$. There are two vertices v and $w \in K_n \times P_1$ such that $r(v|B) = r(w|B)$, a contradiction. Now, we show $\dim(K_n \odot K_1) \le n-1$ by choose a resolving set $B = \{v_1, v_2, \cdots, v_{n-1}\} \subseteq K_n \times P_1$ in $(K_n \times P_1) \odot K_1$. The representation of vertices $v \in (K_n \times P_1) \odot K_1$ by B are

$r(v_{n1}|B) = (1, 1, \cdots, 1)$,
$r(u_{n1}|B) = (2, 2, \cdots, 2)$,
$r(u_{i1}|B) = (\cdots, 2, 1, 2, \cdots)$, for $1 \le i \le n-1$,

vertex u_{i1} is adjacent with v_i and has distance 2 from all other vertices of B. All of those representations are distinct. Therefore, $\dim(K_n \times P_1) \odot K_1) = n-1$.

Case 2. $m = 2$. By contradiction, we will show that $\dim((K_n \times P_2) \odot K_1) \ge n$. Assume that B is a basis of $(K_n \times P_2) \odot K_1$, with $|B| < n$. If $B \subseteq \{v_{11}, v_{21}, \cdots, v_{n1}\}$ or $B \subseteq \{v_{12}, v_{22}, \cdots, v_{n2}\}$, let be the previous one, then there exist $k \in \{1, 2, \cdots, n\}$ such that $r(u_{k1}|B) = \{2, 2, \cdots, 2\} = r(v_{k2}|B)$, a contradiction. Otherwise, there exist $k, l \in \{1, 2, \cdots, n\}$ such that $r(u_{kj}|B) = r(u_{lj}|B)$, for $1 \le j \le 2$, a contradiction too. We will show that $\dim((K_n \times P_2) \odot K_1) \le n$ by choosing a resolving set $B = \{u_{11}, u_{21}, \cdots, u_{n1}\}$. The representation of vertices $v \in (K_n \times P_2) \odot K_1$ by B are

$r(v_{i1}|B) = \{\cdots, 2, 1, 2, \cdots\}$,

v_{i1} is adjacent with u_{i1} and have distance 2 with the others vertex in B,

$r(v_{i2}|B) = \{\cdots, 3, 2, 3, \cdots\}$,
$r(u_{i2}|B) = \{\cdots, 4, 3, 4, \cdots\}$,

It makes all representations of vertices in $(K_n \times P_2) \odot K_1$ are distinct. ∎

Open problem 2. *Find the dimension of* $(K_n \times P_m) \odot K_1$*, for* $n \ge 3$ *and* $m \ge 3$.

Next, we will use the idea of distance similar introduced by Saenpholphat and Zhang in [9] to determine the dimension of corona graph $G \odot \overline{K_m}$, for any graph G and $m \ge 2$. Two vertices u and v of a connected graph G are defined to be *distance similar* if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$. Some of their properties can be found in the following observations.

Observation 1 ([9])**.** *Two vertices* u *and* v *of a connected graph* G *are distance similar if and only if (1)* $uv \notin E(G)$ *and* $N(u) = N(v)$ *or (2)* $uv \in E(G)$ *and* $N[u] = N[v]$.

Observation 2 ([9])**.** *Distance similarity in a connected graph* G *is an equivalence relation on* $V(G)$.

Observation 3 ([9])**.** *If* U *is a distance similar equivalence class of a connected graph* G*, then* U *is either independent in* G *or in* $\overline{G}$.

Observation 4 ([9])**.** *If U is a distance similar equivalence class in a connected graph G with $|U| = p \geq 2$, then every resolving set of G contains at least $p - 1$ vertices from U.*

Theorem 4. *If $G \odot \overline{K_m}$, with $|G| = n$ and $m \geq 2$, $dim(G \odot \overline{K_m}) = n(m-1)$.*

Proof Let $G \odot \overline{K_m}$, where $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $(\overline{K_m})_i : u_{1i}, u_{2i}, \cdots, u_{mi}$ is copy of $\overline{K_m}$ that joining with v_i. Let d_{ij} be the distance between two vertices v_i and v_j in G. For every $i \in \{1, 2, \cdots, n\}$, every pair vertices $u, v \in (\overline{K_m})_i$ holds $d(u, x) = d(v, x)$ for all $x \in V(G \odot \overline{K_m}) - \{u, v\}$. Further, $(\overline{K_m})_i$ is a distance similar equivalence class of $G \odot \overline{K_m}$. By using Observation 2, we have $\dim(G \odot \overline{K_m}) \geq n(m-1)$. Next, we will show that $\dim(G) \leq n(m-1)$. Let $B = \{B_1, B_2, \cdots, B_n\}$, where B_i is a basis of $K_1 \odot (\overline{K_m})_i$. Without loss of generality, let $B_i = \{u_{1i}, u_{2i}, \cdots, u_{(m-1)i}\}$, for every $i \in \{1, 2, \cdots, n\}$. The representation of another vertices in $G \odot \overline{K_m}$ are

$$r(u_{mi}|B) = (\cdots, \underbrace{2, 2, \cdots, 2}_{coord.\ u_{mi}\ by\ B_i}, \cdots),$$

$$r(v_i|B) = (\cdots, \underbrace{1, 1, \cdots, 1}_{coord.\ v_i\ by\ B_i}, \cdots).$$

It makes the representation of every vertex v in G by B is unique. Then B is a resolving set. So, $\dim(G) \leq n(m-1)$. ■

For corona product $G \odot H$, if $G \cong K_1$ and $H \cong \overline{K_m}$, $G \odot H$ is equal to star $\text{Star}(m) = K_1 + \overline{K_m}$. For this graph, if we use Theorem 4 then $\dim(\text{Star}(m)) = m - 1$. This is the same result if we use Theorem A (*iv*) or Theorem A (*v*).

Now, we will determine of a graph with pendant edges more general than corona graphs $G \odot \overline{K_m}$. Let G is a connected graph with order n. Let every vertex v_i of G is joining with m_i number of pendant edges, $m_i \geq 2$ and $1 \leq i \leq n$.

Theorem 5. *For $n \geq 2$,*

$$dim(G) = \sum_{i=1}^{n} (m_i - 1).$$

Proof Similar prove with Theorem 4. ■

Open problem 3. *Find the dimension of $G \odot \overline{K_m}$, with $|G| = n$ and $m \geq 1$.*

References

[1] P.S. Buczkowski, G. Chartrand, C. Poisson, and P. Zhang, On k-dimensional graphs and their bases, *Period. Math. Hungar.*, **46:1** (2003), 9-15.

[2] J. Caceres, C. Hernando, M. Mora, M.L. Puertas, I.M. Pelayo, C. Seara, and D. R. Wood, On the metric dimension of cartesian products of graphs, *Discrete Math.*, submitted.

[3] J. Caceres, C. Hernando, M. Mora, M.L. Puertas, I.M. Pelayo, and C. Seara, On the metric dimension of some families of graphs, *Electronic Notes in Discrete Math.*, **22** (2005), 129-133.

[4] G.G. Chappell, J. Gimbel, and C. Hartman, Bounds on the metric and partition dimensions of a graph, preprint.

[5] G. Chartrand, L. Eroh, M.A. Johnson, and O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, **105** (2000), 99 - 113.

[6] G. Chartrand, and L. Lesniak, *Graphs and Digraphs*, 3rd ed., Chapman and Hall/CRC, 2000.

[7] M. Fehr, S. Gosselin, and O.R. Oellermann, The metric dimension of Cayley digraphs, *Discrete Math.*, **306** (2006), 31-41.

[8] F. Harary and R.A. Melter, On the metric dimension of a graph, *Ars. Combin.*, **2** (1976), 191-195.

[9] V. Saenpholphat and P. Zhang, Some results on connected resolvability in graphs, *Congr. Numer.*, **158** (2002), 5-19.

[10] P.J. Slater, Leaves of trees, *Congr. Numer.*, **14** (1975), 549-559.

[11] P.J. Slater, Dominating and reference sets in graphs, *J. Math. Phys. Sci.*, **22** (1988), 445-455.